CONVEX RATIONALLY CONNECTED VARIETIES

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0. Introduction

Let X be a nonsingular projective variety over $\mathbb{C}$. A morphism,

$$\mu : \mathbf{P}^1 \to X,$$

is *unobstructed* if $H^1(\mathbf{P}^1, \mu^* T_X) = 0$. The variety X is *convex* if all morphisms $\mu : \mathbf{P}^1 \to X$ are unobstructed.

A *rational curve* in X is the image of a morphism

$$\mu : \mathbf{P}^1 \to X.$$

The variety X is *rationally connected* if all pairs of points of X are connected by rational curves.

Homogeneous spaces $\mathbf{G}/\mathbf{P}$ for connected linear algebraic groups are convex, rationally connected, nonsingular, projective varieties. Convexity is a consequence of the global generation of the tangent bundle of $\mathbf{G}/\mathbf{P}$. Rational connectedness is consequence of the rationality of $\mathbf{G}/\mathbf{P}$.

The following speculation arose at dinner after an algebraic geometry seminar at Princeton in the fall of 2002.

Speculation. If X is convex and rationally connected, then X is a homogeneous space.

The failure of the speculation would perhaps be more interesting than the success.

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1. Complete intersections

The only real evidence known to the author is the following result for complete intersections in projective space.

Theorem. *If $X \subset \mathbf{P}^n$ is a convex, rationally connected, nonsingular complete intersection, then X is a homogeneous space.*

Proof. We first consider nonsingular complete intersections of dimension at most 1:

(i) in dimension 0, only points are rationally connected,
(ii) in dimension 1, only $\mathbf{P}^1$ is rationally connected.

Hence, the rationally connected complete intersections of dimension at most 1 are simply connected.

Let $X \subset \mathbf{P}^n$ be a *generic* complete intersection of type $(d_1, \ldots, d_l)$. Let $d = \sum_{i=1}^{l} d_i$. By the results of [4], X is rationally connected if and only if $d \leq n$. Moreover, if X is rationally connected, then X must be simply connected: if the dimension of X at least 2, X is simply connected by Lefschetz, see [5].

Let M denote the parameter space of lines in X. M is a non-empty, nonsingular variety of dimension $2n-2-d-l$. Non-emptiness can be seen by several methods. For example, the nonvanishing in degree 1 of the 1-point series of the quantum cohomology of X implies M is non-empty, see [1], [6]. Nonsingularity is a consequence of the genericity of X. Let $\pi : U \to M$ denote the universal family of lines over M, and let

$$\nu : U \to X$$

denote the universal morphism.

Let L be a line on X. If X is convex, the normal bundle N_L of L in X must be semi-positive. If N_L has a negative line summand, then every double cover of L,

$$\mu : \mathbf{P}^1 \to L \subset X,$$

is obstructed. Since the degree of N_L is $n-d-1$, we may assume $d \leq n-1$.

Every semi-positive bundle on $\mathbf{P}^1$ is generated by global sections. Hence, if every line L has semi-positive normal bundle, we easily conclude the morphism ν is smooth and surjects onto X. The fiber of ν over $x \in X$ is the parameter space of lines passing through x.

We now consider the Leray spectral sequence for the fibration ν, see [2]. The Leray spectral sequence degenerates at the E_2 term,

$$E_2^{pq} = H^p(X, R^q\nu_*\mathbb{C}).$$

Since X is simply connected, all local systems on X are constant. Hence,

$$E_2^{pq} = H^p(X, R^q\nu_*\mathbb{C}) = H^p(X, \mathbb{C}) \otimes H^q(F, \mathbb{C}),$$

where F denotes the fiber of ν.

Let $p_U(t)$, $p_F(t)$, and $p_X(t)$ denote the Poincaré polynomials of the manifolds U, F, and X. We conclude,

$$p_U = p_F \cdot p_X.$$

On the other hand, since U is a locally trivial fibration over M, the polynomial $p_{\mathbf{P}^1}$ must divide p_U. Since

$$p_{\mathbf{P}^1} = 1 + t^2$$

is irreducible over the integers, we find $1+t^2$ divides either p_F or p_X.

We have proven the following result. Let $X \subset \mathbf{P}^n$ be a generic complete intersection of type $(d_1, \dots, d_l)$ satisfying $d \leq n-1$. If every line of X has a semi-positive normal bundle, then either $p_F(i) = 0$ or $p_X(i) = 0$.

Consider the fiber F of ν over x. The dimension of F is $n-1-d$. In fact, F is a complete intersection of type

$$(1, 2, 3, \dots, d_1, 1, 2, 3, \dots, d_2, \dots, 1, 2, 3, \dots, d_l)$$

in the projective space $\mathbf{P}^{n-1}$ of lines of $\mathbf{P}^n$ passing through x.

If $p_F(i) = 0$, then the type of F must be one of the three types allowed by the Lemma below. If $p_X(i) = 0$, then the type of X must be one of the three allowed by the Lemma. Since, one of the two polynomial evaluations must

vanish, we conclude the type of X must be either $(1, \dots, 1)$ or $(1, \dots, 1, 2)$. Clearly both are types of homogeneous varieties.

If X is not of one of the two above types, then X must contain a line L for which N_L has a negative line summand. Since X was assumed to be general, every nonsingular complete intersection Y of the type of X must also contain such a line by taking a limit of L.

Therefore, if the type of a nonsingular complete intersection Y is not $(1, \dots, 1)$ or $(1, \dots, 1, 2)$, then Y is not a convex, rationally connected variety. □

The proof of the Theorem also shows homogeneous complete intersections in projective space must be of type $(1, \dots, 1)$ or $(1, \dots, 1, 2)$.

Lemma. *Let $Y \subset \mathbf{P}^n$ be a nonsingular complete intersection of dimension k. Let $p_Y(t)$ be the Poincaré polynomial of Y. If $p_Y(i) = 0$, then one of the following three possibilities hold:*

(i) *the type of Y is $(1, \dots, 1)$ and k is odd,*
(ii) *the type of Y is $(1, \dots, 1, 2)$ and k is odd,*
(iii) *the type of Y is $(1, \dots, 1, 2)$, and $k = 2 \mod 4$.*

Proof. Let $Y \subset \mathbf{P}^n$ be a nonsingular complete intersection of dimension k. The cohomology of Y is determined by the Lefschetz isomorphism except in the middle (real) dimension k. The cohomology determined by Lefschetz is of rank 1 in all even (real) dimensions. If k is odd, then

$$p_Y(t) = \sum_{q=0}^{k} t^{2q} + b_k t^k,$$

where b_k is the k^{th} Betti number. We see $p_Y(i) = 0$ if and only if $b_k = 0$. If k is even,

$$p_Y(t) = \sum_{q=0}^{k} t^{2q} + (b_k - 1)t^k.$$

We see $p_Y(i) = 0$ if and only if $k = 2 \mod 4$ and $b_k - 1 = 1$.

Assume $p_Y(i) = 0$. Let $(e_1, \ldots, e_k)$ be the type of Y. Let e be the largest element of the type.

Let $Z \subset \mathbf{P}^n$ be a nonsingular projective variety of dimension r. Let $H_d \subset \mathbf{P}^n$ be a general hypersurface of degree d. The dimension,

$$h^{r-1}(Z \cap H_d, \mathbb{C}),$$

is a non-decreasing function of d, see [3]. Hence, we can bound b_k for Y from below by the middle cohomology b'_k of the complete intersection $Y' \subset \mathbf{P}^n$ of type $(e, 1, \ldots, 1)$,

$$b_k \geq b'_k.$$

The variety Y' may then be viewed as a hypersurface of degree e in the smaller projective space $\mathbf{P}^{k+1}$.

For a hypersurface $Y' \subset \mathbf{P}^{k+1}$ of degree e, the middle cohomology b'_k is given by the following formula:

$$b'_k - \delta_k = \frac{(e-1)}{e}((e-1)^{k+1} - (-1)^{k+1}),$$

where δ_k is 1 if k is even and 0 if k is odd. If k is odd,

$$b'_k = \frac{(e-1)}{e}((e-1)^{k+1} - 1).$$

Then, $b'_k > 0$ if $e > 2$. If k is even,

$$b'_k - 1 = \frac{(e-1)}{e}((e-1)^{k+1} + 1).$$

Then, $b_k - 1 > 1$ if $e > 2$. Therefore, we conclude $e \leq 2$.

If $e = 1$, then case (i) of the Lemma is obtained. It is easy to check k must be odd for $p_Y(i) = 0$ to hold.

Let $e = 2$. If Y is of type $(1, \ldots, 1, 2)$, then either case (ii) or (iii) of the Lemma is obtained. If k is even,

$$k = 2 \bmod 4$$

must be satisfied in order for $p_Y(i) = 0$ to hold.

If Y is not of type $(1, \ldots, 1, 2)$, then the next largest type of Y must be at least 2. As before, we may bound b_k from below by the middle cohomology

b'_k of the complete intersection of type $(2,2)$ in $\mathbf{P}^{k+2}$. If k is odd, the calculation below shows

$$b'_k = k+1 > 0.$$

If k is even, the calculation below shows

$$b'_k - 1 = k+3 > 1.$$

In fact, the type of Y *can not* contain two elements greater than 1 in $p_Y(i) = 0$.

Let $k \geq 0$. The Euler characteristic $\chi_{22}(k)$ of a nonsingular complete intersection of type $(2,2)$ in $\mathbf{P}^{k+2}$ is:

$$\int_{\mathbf{P}^{k+2}} (\frac{2H}{1+2H})^2 (1+H)^{k+3} \quad = \quad \sum_{i=0}^{k} 2^{k+2-i}(-1)^{k-i}(k+1-i)\binom{k+3}{i}.$$

On the other hand, since

$$(k+3)(t-1)^{k+2} - (t-1)^{k+3} + (-1)^{k+3} = \sum_{i=0}^{k+2} t^{k+2-i}(-1)^i(k+3-i-t)\binom{k+3}{i},$$

we find:

$$(-1)^k \chi_{22}(k) = k+2+(-1)^k(k+2).$$

The formulas for b'_k then follow easily. □

2. Homogeneous complete intersections

It is interesting to see how the homogeneous complete intersections survive the above analysis.

First, consider a complete intersection $X \subset \mathbf{P}^n$ of type $(1, \dots, 1)$. Then, F is of dimension $n-1-l$, and X is a dimension $n-l$. Both are complete intersections of hyperplanes. Since one of $n-1-l$ and $n-l$ is odd, exactly one of the conditions $p_F(i) = 0$ or $p_X(i) = 0$ holds by part (i) of the Lemma.

Next, consider a complete intersection $X \subset \mathbf{P}^n$ of type $(1, \dots, 1, 2)$ where $l+1 \leq n-1$ Then, F is of dimension $n-1-l-1$, and X is of dimension $n-l$. There are two cases:

(i) If $n-l-2$ and $n-l$ are odd, then both $p_F(i)=0$ and $p_X(i)=0$ by part (ii) of the Lemma.

(ii) If $n-l-2$ and $n-l$ are even, then one of $(n-l-2)/2$ and $(n-l)/2$ is odd. Hence, exactly one of the conditions $p_F(i)=0$ and $p_X(i)=0$ holds by part (iii) of the Lemma.

References

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